The Technical University of Denmark

Written (trial) exam, May 3, 2024
Course: 01004 Mathematics 1b
Aids: All aids allowed by DTU (mobile phones and internet access not allowed)
Duration: 4 hours
Weights: Ex. 1: 15%, Ex. 2: 20%, Ex. 3: 15%, Ex. 4: 20%, Ex. 5: 15%, Ex. 6: 15%.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, etc.) are to the lecture notes.

Exercise 1. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, the first-order partial derivatives are provided:

$$\frac{\partial f}{\partial x}(x,y) = 6x - 6y \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 6y^2 - 6x.$$

(a) Determine all stationary points of f.

(b) Compute the second-order partial derivatives of f and find the Hessian matrix $\boldsymbol{H}_f(x,y)$ of f. Determine the points in the (x,y) plane where f has a local maximum, a local minimum or a saddle point.

(c) It is now stated that $f(0,0) = 1$. Determine the second-degree Taylor polynomial $P_2(x,y)$ for f with the expansion point $(0,0)$.

Exercise 2. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

(a) Find the third-degree Taylor polynomial $P_3(x)$ of $\sin(x)$ with expansion point $x_0 = 0$.

(b) Show that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Hint: Use (a) and Taylor's limit formula.

(c) Argue that f is continuous on $\mathbb{R}$.

(d) Compute, e.g. using SymPy, a decimal approximation of $\int_0^1 f(x)\,\mathrm{d}x$. You should include at least 5 decimals.

(e) Compute a Riemann sum $S_J = \sum_{j=1}^{J} f(\xi_j)\Delta x_j$ approximating $\int_0^1 f(x)\,\mathrm{d}x$, where we require that $\Delta x_j \leq 1/30$ for each $j = 1, \dots, J$.

(f) Compute $\int_0^1 P_3(x)\,\mathrm{d}x$. Is it a better or worse approximation of $\int_0^1 f(x)\,\mathrm{d}x$ than the Riemann sum from the previous question.

The set of problems CONTINUES.

Exercise 3. Let $t \in \mathbb{R}$ and define $C_t \in \mathsf{M}_4(\mathbb{R})$ by

$$C_t = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ t & 3 & 4 & 1 \end{bmatrix}.$$

(a) Show that the matrix C_t is normal if and only if $t = 2$.

Let $A = C_2$ (i.e., $t = 2$). It is stated that $\boldsymbol{v}_1 = [1, 1, 1, 1]^T$ and $\boldsymbol{v}_2 = [1, i, -1, -i]^T$ are eigenvectors of A.

(b) Find the eigenvalue λ_1 associated with the eigenvector $\boldsymbol{v}_1$.

(c) Find the eigenvalue λ_2 associated with the eigenvector $\boldsymbol{v}_2$.

(d) Argue that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal.

(e) Compute the norm of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Is the list $\boldsymbol{v}_1, \boldsymbol{v}_2$ orthonormal?

Exercise 4. A quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$q(x_1, x_2) = 2{x_1}^2 - 2x_1x_2 + 2{x_2}^2 - 4x_1 + 2x_2 + 2.$$

(a) State $A \in \mathbb{R}^{2\times 2}$, $\boldsymbol{b} \in \mathbb{R}^2$ and $c \in \mathbb{R}$ so that $q(x_1, x_2) = \boldsymbol{x}^T A\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{b} + c$, where $\boldsymbol{x} = [x_1, x_2]^T$.

(b) Find an orthogonal (change-of-basis) matrix $\boldsymbol{Q}$ that reduces the quadratic form q such that it in the new coordinates $(\tilde{x}_1, \tilde{x}_2)$ does not contain "mixed terms", where

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \boldsymbol{Q}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

If one uses a change of basis as described in the previous question, q can be written in the reduced form

$$q_1(\tilde{x}_1, \tilde{x}_2) = \alpha(\tilde{x}_1 - \gamma)^2 + \beta(\tilde{x}_2 - \delta)^2.$$

(c) Determine the real numbers α, β, γ and δ that fulfill this.

(d) The function q_1 has one stationary point located at (γ, δ) in $(\tilde{x}_1, \tilde{x}_2)$ coordinates. What is the location of the stationary point in (x_1, x_2) coordinates? Explain why q has a *local minimum* at the stationary point.

The set of problems CONTINUES.

Exercise 5. A (solid) region $\Omega \subset \mathbb{R}^3$ is given by the parametric representation

$$\boldsymbol{r}(u, v, w) = \begin{bmatrix} v\,u^2\,\cos(w) \\ v\,u^2\,\sin(w) \\ u \end{bmatrix}, \quad u \in [0,1], v \in [0,1], w \in \left[0, \frac{\pi}{2}\right],$$

that is, $\Omega = \{\, \boldsymbol{r}(u,v,w) \mid u \in [0,1], v \in [0,1], w \in \left[0, \frac{\pi}{2}\right] \,\}$.

(a) Plot the region Ω. Determine the Jacobian matrix and Jacobian determinant of $\boldsymbol{r}$.

Consider the C^∞ vector field $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{V}(x,y,z) = (x + \mathrm{e}^{yz}, 2y - \mathrm{e}^{xz}, 3z + \mathrm{e}^{xy})$. Define the function

$$f : \mathbb{R}^3 \to \mathbb{R},\ f = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

(b) Find an expression of $f(x,y,z)$.

(c) Argue that f is Riemann integrable over Ω.

(d) Determine the Riemann integral $\int_\Omega f(x,y,z)\,\mathrm{d}(x,y,z)$.

Exercise 6. Define the rectangle $\Gamma \subset \mathbb{R}^2$ by

$$\Gamma = \{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}.$$

A function $h : \Gamma \to \mathbb{R}$ is given by $h(x,y) = 2x - y + 1$. Let G denote the graph of h, i.e., $G = \{\, (x,y,h(x,y)) \in \mathbb{R}^3 \mid (x,y) \in \Gamma \,\}$.

(a) Determine the surface area of G.

The line segment between the points $(0,1)$ and $(2,0)$ divides Γ into two parts. Let Γ_1 denote the "lower part", and G_1 denote the part of the graph of h that lies "vertically above Γ_1", i.e., $G_1 = \{\, (x,y,h(x,y)) \in \mathbb{R}^3 \mid (x,y) \in \Gamma_1 \,\}$.

(b) Find a parametrization of G_1, and determine the associated Jacobian function.

(c) Determine the surface integral of f over G_1, where the function f is defined by

$$f(x,y,z) = x + y + z - 1, \quad (x,y,z) \in \mathbb{R}^3.$$

The set of problems is completed.