# Danmarks Tekniske Universitet

Written exam, August 22, 2024
Course name: Mathematics 1b
Course number: 01004
Aids: All aids allowed by DTU (without internet)
Duration: 4 hours
Weights: Ex. 1: 25%, Ex. 2: 25%, Ex. 3: 25%, Ex. 4: 25%.

*In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, etc.) are to the lecture notes.*

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**Exercise 1.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = 2x^2 + y^2 - xy^2.$$

(a) Plot the level set $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 2\}$. A straight line is part of this level set. Describe the straight line (either by stating the equation of the line or by a parametrization of the line).

(b) Compute the gradient $\nabla f(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

(c) Compute the Hessian matrix $\boldsymbol{H}_f(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

(d) Show that $(0, 0)$ is a stationary point of $f$. Determine using the Hessian matrix $\boldsymbol{H}_f(0, 0)$ whether it is a local minimum, a local maximum or a saddle point.

(e) Find all stationary points of $f$. Determine using the Hessian matrix whether, for each stationary point, it is a local minimum, a local maximum or a saddle point.

*The set of problems CONTINUES.*

**Exercise 2.** Let $\boldsymbol{y} = [1, 2, 2, 4]^T$ be a (column) vector in $\mathbb{R}^4$ equipped with the standard inner product $\langle\cdot,\cdot\rangle$. Define $Y = \operatorname{span}\{\boldsymbol{y}\}$. Define the matrix $A \in \mathbb{R}^{4\times 4}$ by

$$A = \boldsymbol{y}\boldsymbol{y}^T.$$

(a) Show that $A$ is symmetric.

(b) Argue that the columns of $A$ are scalar multiple of each other and that the rank of $A$ is one.

(c) Give an example of a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4$ that satisfies $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$. Recall that $\boldsymbol{y}$ is defined above.

(d) Let $\boldsymbol{x} \in \mathbb{R}^4$ be any vector that satisfies $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$. Show that $A\boldsymbol{x} = \boldsymbol{0}$.

It can be shown that $Y^\perp = \ker A$, where $Y^\perp$ is the orthogonal complement of $Y$ and $\ker A$ is the kernel (also called null space) of the matrix $A$. You may use this fact without proof.

(e) Argue that $\dim(Y^\perp) = 3$.

(f) Find an orthonormal basis for $Y^\perp$.

**Exercise 3.** Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

(a) Plot the graph of the function $f$. State the function value $f(k\pi)$ for each integer $k \in \mathbb{Z}$.

(b) Find the degree-two Taylor polynomial $P_2(x)$ of $\sin(x)$ at $x_0 = 0$.

(c) Show that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

*Hint: Use (b) and Taylor's limit formula.*

(d) Argue that $f$ is continuous.

Let the function $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} \frac{\sin(3x)}{x} & \text{for } x \neq 0, \\ c & \text{for } x = 0, \end{cases}$$

where $c \in \mathbb{R}$.

(e) Specify a value of $c$ that makes $g$ continuous.

*The set of problems CONTINUES.*

**Exercise 4.** Consider the subset $A \subset \mathbb{R}^2$ given by:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 \leq x_1^2 + x_2^2 \leq 4 \wedge x_1 \geq 0 \wedge x_2 \geq 0\}.$$

Define the function $f : A \to \mathbb{R}$ by

$$f(x_1, x_2) = \ln(x_1^2 + x_2^2).$$

(a) Let $r > 0$. Show that $r^2 \ln(r^2) - r^2$ is an anti-derivative of $2\ln(r^2)r$, i.e., show that

$$\int 2\ln(r^2)r \, \mathrm{d}r = r^2 \ln(r^2) - r^2.$$

(b) Find all anti-derivatives of $2\ln(r^2)r$, where $r > 0$.

(c) Find a parametrization of $A$ in polar coordinates. State the Jacobian matrix and determinant of the parametrization.

(d) Argue that $f$ is Riemann integrable.

(e) Compute the integral $\int_A f(x_1, x_2) \, \mathrm{d}(x_1, x_2)$.

*The set of problems is completed.*