Danmarks Tekniske Universitet

Written (test) exam, May 2 or 3, 2024
Course: 01002 Mathematics 1b
Aids: All aids allowed by DTU (mobile phones and internet access not allowed)
Duration: 4 hours
Weights: Ex. 1: 15%, Ex. 2: 20%, Ex. 3: 15%, Ex. 4: 20%, Ex. 5: 15%, Ex. 6: 15%.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, etc.) are to the lecture notes. A Danish version of the exam set follows *after* the English version.

Exercise 1. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, the first-order partial derivatives are provided:

$$\frac{\partial f}{\partial x}(x,y) = 6x - 6y \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 6y^2 - 6x.$$

(a) Determine all stationary points of f.

(b) Compute the second-order partial derivatives of f and find the Hessian matrix $\boldsymbol{H}_f(x,y)$ of f at (x,y). Determine the points in the (x,y)-plane where f has a local maximum, a local minimum or a saddle point.

(c) It is now stated that $f(0,0) = 1$. Determine the second-degree Taylor polynomial $P_2(x,y)$ for f with the expansion point $(0,0)$.

Exercise 2. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

(a) Find the third-degree Taylor polynomial $P_3(x)$ of $\sin(x)$ at $x_0 = 0$.

(b) Show that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Hint: Use (a) and Taylor's limit formula.

(c) Argue that f is continuous on $\mathbb{R}$.

(d) Compute, e.g., using SymPy, a decimal approximation of $\int_0^1 f(x)\,\mathrm{d}x$. You should include at least 5 decimals.

(e) Compute a Riemann sum $S_J = \sum_{j=1}^{J} f(\xi_j)\Delta x_j$ approximating $\int_0^1 f(x)\,\mathrm{d}x$, where we require that $\Delta x_j \leq 1/30$ for each $j = 1, \ldots, J$.

(f) Compute $\int_0^1 P_3(x)\,\mathrm{d}x$. Is it a better or worse approximation of $\int_0^1 f(x)\,\mathrm{d}x$ than the Riemann sum from the previous question?

The set of problems CONTINUES.

Exercise 3. Let $t \in \mathbb{R}$ and define $C_t \in \mathsf{M}_4(\mathbb{R})$ by

$$C_t = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ t & 3 & 4 & 1 \end{bmatrix}.$$

(a) Show that the matrix C_t is normal if and only if $t = 2$.

Let $A = C_2$ (i.e., $t = 2$). It is stated that $\boldsymbol{v}_1 = [1, 1, 1, 1]^T$ and $\boldsymbol{v}_2 = [1, i, -1, -i]^T$ are eigenvectors of A.

(b) Find the eigenvalue λ_1 associated with the eigenvector $\boldsymbol{v}_1$.

(c) Find the eigenvalue λ_2 associated with the eigenvector $\boldsymbol{v}_2$.

(d) Argue that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal.

(e) Compute the norm of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Is the list $\boldsymbol{v}_1, \boldsymbol{v}_2$ orthonormal?

Exercise 4. A quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$q(x_1, x_2) = 2{x_1}^2 - 2x_1x_2 + 2{x_2}^2 - 4x_1 + 2x_2 + 2.$$

(a) State $A \in \mathbb{R}^{2\times 2}$, $\boldsymbol{b} \in \mathbb{R}^2$ and $c \in \mathbb{R}$ so that $q(x_1, x_2) = \boldsymbol{x}^T A\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{b} + c$, where $\boldsymbol{x} = [x_1, x_2]^T$.

(b) Find an orthogonal (change-of-basis) matrix $\boldsymbol{Q}$ that reduces the quadratic form q such that it in the new coordinates $(\tilde{x}_1, \tilde{x}_2)$ does not contain "mixed terms", where

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \boldsymbol{Q}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

If one uses a change of basis as described in the previous question, q can be written in the reduced form

$$q_1(\tilde{x}_1, \tilde{x}_2) = \alpha(\tilde{x}_1 - \gamma)^2 + \beta(\tilde{x}_2 - \delta)^2.$$

(c) Determine the real numbers α, β, γ and δ that fulfill this.

(d) The function q_1 has one stationary point located at (γ, δ) in $(\tilde{x}_1, \tilde{x}_2)$-coordinates. What is the location of the stationary point in (x_1, x_2)-coordinates? Explain why q has a *local minimum* at the stationary point.

The set of problems CONTINUES.

Exercise 5. A (solid) region $\Omega \subset \mathbb{R}^3$ is given by the parametric representation

$$\boldsymbol{r}(u,v,w) = \begin{bmatrix} v\,u^2\,\cos(w) \\ v\,u^2\,\sin(w) \\ u \end{bmatrix}, \quad u \in [0,1], v \in [0,1], w \in \left[0, \frac{\pi}{2}\right],$$

that is, $\Omega = \{\, \boldsymbol{r}(u,v,w) \mid u \in [0,1], v \in [0,1], w \in \left[0, \frac{\pi}{2}\right] \,\}$.

(a) Plot the region Ω. Determine the Jacobian matrix and Jacobian determinant of $\boldsymbol{r}$.

Consider the C^∞ vector field $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{V}(x,y,z) = (x + \mathrm{e}^{yz}, 2y - \mathrm{e}^{xz}, 3z + \mathrm{e}^{xy})$. Define the function

$$f : \mathbb{R}^3 \to \mathbb{R},\; f = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

(b) Find an expression of $f(x,y,z)$.

(c) Argue that f is Riemann integrable over Ω.

(d) Determine the Riemann integral $\int_\Omega f(x,y,z)\,\mathrm{d}(x,y,z)$.

Exercise 6. Define the rectangle $\Gamma \subset \mathbb{R}^2$ by

$$\Gamma = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 1\}.$$

A function $h : \Gamma \to \mathbb{R}$ is given by $h(x,y) = 2x - y + 1$. Let G denote the graph of h, i.e., $G = \{\, (x,y,h(x,y)) \in \mathbb{R}^3 \mid (x,y) \in \Gamma \,\}$.

(a) Determine the surface area of G.

The line segment between the points $(0,1)$ and $(2,0)$ divides Γ into two parts. Let Γ_1 denote the "lower part", and G_1 denote the part of the graph of h that lies "vertically above Γ_1", i.e., $G_1 = \{\, (x,y,h(x,y)) \in \mathbb{R}^3 \mid (x,y) \in \Gamma_1 \,\}$.

(b) Find a parametrization of G_1, and determine the associated Jacobian function.

(c) Determine the surface integral of f over G_1, where the function f is defined by

$$f(x,y,z) = x + y + z - 1, \quad (x,y,z) \in \mathbb{R}^3.$$

The set of problems is completed.

Opgave 1. En funktion $f : \mathbb{R}^2 \to \mathbb{R}$ har følgende førsteordens partielle afledede:

$$\frac{\partial f}{\partial x}(x, y) = 6x - 6y \quad \text{og} \quad \frac{\partial f}{\partial y}(x, y) = 6y^2 - 6x.$$

(a) Bestem alle stationære punkter for f.

(b) Udregn de andenordens partielle afledede af f og find Hesse-matricen $\boldsymbol{H}_f(x, y)$. Bestem punkterne i (x, y)-planen, hvori f har et lokalt maksimum, et lokalt minimum eller et sadelpunkt.

(c) Det angives nu, at $f(0, 0) = 1$. Bestem Taylor-polynomiet af anden grad $P_2(x, y)$ for f med udviklingspunkt $(0, 0)$.

Opgave 2. Definer funktionen $f : \mathbb{R} \to \mathbb{R}$ ved

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

(a) Find tredjegrads Taylor-polynomiet $P_3(x)$ af $\sin(x)$ med udviklingspunkt $x_0 = 0$.

(b) Vis at

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Hint: Brug (a) og Taylors grænseformel.

(c) Argumenter for at f er kontinuert på hele $\mathbb{R}$.

(d) Beregn, fx ved hjælp af SymPy, en decimaltals tilnærmelse af $\int_0^1 f(x)\,\mathrm{d}x$. Du skal angive mindst 5 betydende cifre.

(e) Udregn en Riemann-sum $S_J = \sum_{j=1}^{J} f(\xi_j)\Delta x_j$, der approksimerer $\int_0^1 f(x)\,\mathrm{d}x$, hvor det kræves, at $\Delta x_j \leq 1/30$ for hvert $j = 1, \ldots, J$.

(f) Udregn $\int_0^1 P_3(x)\,\mathrm{d}x$. Er det en bedre eller dårligere approksimation af $\int_0^1 f(x)\,\mathrm{d}x$ end Riemann-summen fra det foregående spørgsmål?

Opgavesættet FORTSÆTTER.

Opgave 3. Lad $t \in \mathbb{R}$ og definer $C_t \in \mathsf{M}_4(\mathbb{R})$ ved

$$C_t = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ t & 3 & 4 & 1 \end{bmatrix}.$$

(a) Vis at matricen C_t er normal, hvis og kun hvis $t = 2$.

Lad $A = C_2$ (dvs. $t = 2$). Det er angivet, at $\boldsymbol{v}_1 = [1, 1, 1, 1]^T$ og $\boldsymbol{v}_2 = [1, i, -1, -i]^T$ er egenvektorer af A.

(b) Find egenværdien λ_1 tilhørende egenvektoren $\boldsymbol{v}_1$.

(c) Find egenværdien λ_2 tilhørende egenvektoren $\boldsymbol{v}_2$.

(d) Argumenter for at $\boldsymbol{v}_1$ og $\boldsymbol{v}_2$ er ortogonale.

(e) Beregn normen for $\boldsymbol{v}_1$ og $\boldsymbol{v}_2$. Er listen $\boldsymbol{v}_1, \boldsymbol{v}_2$ ortonormal?

Opgave 4. En kvadratisk form $q : \mathbb{R}^2 \to \mathbb{R}$ er givet ved

$$q(x_1, x_2) = 2{x_1}^2 - 2x_1x_2 + 2{x_2}^2 - 4x_1 + 2x_2 + 2.$$

(a) Angiv $A \in \mathbb{R}^{2\times 2}$, $\boldsymbol{b} \in \mathbb{R}^2$ og $c \in \mathbb{R}$, således at $q(x_1, x_2) = \boldsymbol{x}^T A\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{b} + c$, hvor $\boldsymbol{x} = [x_1, x_2]^T$.

(b) Find en ortogonal basisskifte-matrix $\boldsymbol{Q}$, der reducerer den kvadratiske form q således, at den i de nye koordinater $(\tilde{x}_1, \tilde{x}_2)$ ikke indeholder "blandede led", hvor

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \boldsymbol{Q}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hvis man bruger et basisskifte som beskrevet i det foregående spørgsmål, kan q skrives i reduceret form:

$$q_1(\tilde{x}_1, \tilde{x}_2) = \alpha(\tilde{x}_1 - \gamma)^2 + \beta(\tilde{x}_2 - \delta)^2.$$

(c) Bestem de reelle tal α, β, γ og δ, der opfylder dette.

(d) Funktionen q_1 har et stationært punkt i (γ, δ) i $(\tilde{x}_1, \tilde{x}_2)$-koordinater. Hvad er placeringen af det stationære punkt i (x_1, x_2)-koordinater? Forklar hvorfor q har et *lokalt minimum* ved det stationære punkt.

Opgavesættet FORTSÆTTER.

Opgave 5. Et område $\Omega \subset \mathbb{R}^3$ er givet ved parameterfremstillingen:

$$\boldsymbol{r}(u,v,w) = \begin{bmatrix} v\,u^2\,\cos(w) \\ v\,u^2\,\sin(w) \\ u \end{bmatrix}, \quad u \in [0,1], v \in [0,1], w \in \left[0, \frac{\pi}{2}\right],$$

dvs. $\Omega = \{\, \boldsymbol{r}(u,v,w) \mid u \in [0,1], v \in [0,1], w \in \left[0, \frac{\pi}{2}\right] \,\}$.

(a) Plot området Ω. Bestem Jacobi-matricen og Jacobi-determinanten for $\boldsymbol{r}$.

Betragt C^∞ vektorfeltet $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ givet ved $\boldsymbol{V}(x,y,z) = (x + \mathrm{e}^{yz}, 2y - \mathrm{e}^{xz}, 3z + \mathrm{e}^{xy})$. Definer funktionen

$$f : \mathbb{R}^3 \to \mathbb{R},\ f = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

(b) Find et udtryk for $f(x,y,z)$.

(c) Argumenter for at f er Riemann-integrabel over Ω.

(d) Bestem Riemann-integralet $\int_\Omega f(x,y,z)\,\mathrm{d}(x,y,z)$.

Opgave 6. Betragt rektanglet $\Gamma \subset \mathbb{R}^2$ givet ved

$$\Gamma = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 1\}.$$

En funktion $h : \Gamma \to \mathbb{R}$ er givet ved $h(x,y) = 2x - y + 1$. Lad G betegne grafen for h, dvs. $G = \{\, (x,y,h(x,y)) \in \mathbb{R}^3 \mid (x,y) \in \Gamma \,\}$.

(a) Bestem arealet af G.

Linjestykket mellem punkterne $(0,1)$ og $(2,0)$ deler Γ i to dele. Lad Γ_1 betegne den "nederste del", og lad G_1 betegne den del af grafen for h, der ligger "lodret over Γ_1", dvs. $G_1 = \{\, (x,y,h(x,y)) \in \mathbb{R}^3 \mid (x,y) \in \Gamma_1 \,\}$.

(b) Find en parametrisering af G_1, og bestem den tilhørende Jacobi-funktion.

(c) Bestem fladeintegralet af f over G_1, hvor funktionen f er defineret ved

$$f(x,y,z) = x + y + z - 1, \quad (x,y,z) \in \mathbb{R}^3.$$

Opgavesættet er afsluttet.